Resit Exam - Statistics (WBMA009-05) 2024/2025

Date and time: January 31, 2025, 15.00-17.00h
Place: BB 5161.0165

Rules to follow:

- This is a closed book resit exam. Consultation of books and notes is **not** permitted. You can use a simple (non-programmable) calculator.
- Write your name and student number onto each paper sheet.
- There are 4 exercises and you can reach 36 points.
- Your resit exam grade will be computed as follows:

$$\text{grade} = (\text{number of points} + 4) \cdot 0.25$$

  rounded to the nearest 0.5, with the usual exception that 5.5 is not possible. Grades below 5.5 are rounded down to 5.0, while grades of 5.5 or higher are rounded up to 6.0.
- ALWAYS include the relevant equation(s) and/or short derivations.
- **We wish you success with the completion of the resit exam!**

START OF RESIT EXAM

1. **Point estimation and sufficiency.** **[6]**
   Let $X_1, \ldots, X_n$ be an i.i.d. sample from a uniform distribution on the interval $[0, \theta]$, where $\theta > 0$ is an unknown parameter. The density of this distribution is:

$$f_\theta(x) = \frac{1}{\theta} \qquad (0 \leq x \leq \theta)$$

   (a) Derive the method of moments (MOM) estimator for θ. **[2]**
   (b) Derive the maximum likelihood (ML) estimator for θ. **[2]**
   (c) Check for both estimators whether they are sufficient. **[2]**

2. **Statistical hypothesis testing** **[8]**
Consider a random sample of size $n = 3$ from the Bernoulli distribution

$$X_1, X_2, X_3 \sim \text{BER}(\theta)$$

with parameter $\theta \in (0, 1)$. Recall that the Bernoulli density is:

$$f_\theta(x) = \theta^x \cdot (1 - \theta)^{1-x} \qquad (x = 0, 1)$$

Consider the simple test problem

$$H_0 : \theta = 0.6 \;\; \textbf{vs.} \;\; H_1 : \theta = 0.2$$

and a test T which rejects the null hypothesis if $X_1 + X_2 + X_3 \leq 1$.

(a) What is the test level α of T? **[2]**
(b) What is the power β of the test T? **[2]**
(c) Show that T is the UMP test. **[4]**

3. **Sample from Gamma distribution.** **[8]**
Consider a random sample from a Gamma distribution

$$X_1, \ldots, X_n \sim \text{GAM}(\alpha, \beta)$$

where $\alpha > 0$ is a known parameter and $\beta > 0$ is an unknown parameter.

The density of the Gamma distribution is:

$$f_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot \exp\{-x\beta\} \qquad (x > 0)$$

(a) Show that the ML estimator of β is $\hat{\beta}_{ML} = \alpha / \bar{X}$. **[1]**
(Check via the 2nd derivative whether you really have a maximum.) **[1]**
(b) Compute the expected Fisher information (for $n = 1$). **[2]**

Suppose that $\alpha = 4$, $n = 81$, and that $\bar{x} = 2$ has been observed.

(c) Make use of the asymptotic normality of the ML estimator to compute a two-sided 80% asymptotic confidence interval for β. **[4]**

HINT: You can assume that all regularity conditions are fulfilled.
See Table 1 on the next page for the relevant quantiles.

α	0.5	0.75	0.9	0.95	0.975	0.99	0.99997
q_α	0	0.7	1.3	1.6	2	2.3	4

Table 1: Approximate quantiles q_α of the $\mathcal{N}(0,1)$ distribution.

4. **Random sample.** **14**

Consider a distribution $\mathcal{D}(\theta_1, \theta_2)$ that depends on two parameters

$$0 < \theta_1 < 1 \quad \text{and} \quad \theta_2 > 0$$

and whose density is given by:

$$f_{\theta_1,\theta_2}(x) = \frac{1-\theta_1}{\theta_2} \cdot \exp\left\{\frac{x}{\theta_2}\right\} \cdot I_{(-\infty,0)}(x) + \frac{\theta_1}{\theta_2} \cdot \exp\left\{-\frac{x}{\theta_2}\right\} \cdot I_{[0,\infty)}(x)$$

We consider a random sample of size n:

$$X_1, \dots, X_n \sim \mathcal{D}(\theta_1, \theta_2)$$

The likelihood is then given by:

$$L(\theta_1, \theta_2) = \left(\frac{1-\theta_1}{\theta_2}\right)^{n-K} \cdot \left(\frac{\theta_1}{\theta_2}\right)^{K} \cdot \exp\left\{\frac{-S}{\theta_2}\right\}$$

where $S := \sum_{i=1}^{n} |X_i|$ and K is the number of non-negative random variables, so that

$$X_{(1)} \leq \dots \leq X_{(n-K)} < 0 \leq X_{(n-K+1)} \leq \dots < X_{(n)}$$

We assume that θ_1 is a known parameter.

(a) Show that S is a sufficient statistic for θ_2. **1**

(b) Show that the ML estimator of θ_2 is $\hat{\theta}_{2,ML} = S/n$. **2**
(Check via the 2nd derivative whether you really have a maximum.) **1**

(c) Show that $E[S] = n \cdot \theta_2$. **2**
HINT: Use that for all $\theta_2 > 0$: $\int_0^\infty \frac{x}{\theta_2} \cdot \exp\left\{-\frac{x}{\theta_2}\right\}\, dx = \theta_2$.

(d) Compute the Fisher information of a sample of size $n = 1$. **2**

(e) Consider the test problem:

$$H_0 : \theta_2 = 0.5 \;\; \textbf{vs.} \;\; H_1 : \theta_2 \neq 0.5$$

Use the asymptotic normality of the ML estimator to construct an asymptotic level α test for this problem. When does the test reject H_0? **2**

(f) Assume that $n = 25$ and that $\hat{\theta}_{2,ML} = 0.37$ has been observed.
Check whether the asymptotic test from part (e) can reject the null hypothesis to the test level $\alpha = 0.1$ and give the p-value of the test. **2+2**

HINT: In parts (e-f) You can assume that all regularity conditions are fulfilled. See Table 1 above for the relevant quantiles.